Phys 410

Fall 2015

Homework #7

Due Thursday, 29 October, 2015

All problems are from Taylor, *Classical Mechanics*.

1) Problem 7.31 Pendulum with oscillating point of suspension
2) Problem 7.34 Harmonic oscillation with spring of non-zero mass
3) Problem 7.35 Rotating loop with bead
4) Problem 7.37 Rotating mass on table
5) Problem 7.41 Bead on a rotating parabola
6) Problem 8.1 Two-particle problem
7) Problem 8.3 Two particles interacting by means of a spring
8) Problem 8.6 Angular momentum
9) Problem 8.9 Two-dimensional harmonic motion
10) Problem 8.12 Planetary angular momentum
11) Is the motion a minimum in the action or a saddle point? Consider the case of a simple harmonic oscillator.

a) Write down the Lagrangian for a one-dimensional simple harmonic oscillator (mass on a spring, no gravity, no friction) in terms of the coordinate x, it's time derivative $\dot{x}$, m and k, where m is the mass and k is the spring constant. Let $x_0(t)$ be the true path of the oscillator, so that $x_0(t)$ satisfies the SHO equation of motion. We now consider variations on this path of the form $x_0(t) + \xi(t)$, where $\xi(t)$ goes to zero at $t = 0$ and $t = t_1$. If $S[\xi]$ represents that action for the variation ξ, show that

$$S[\xi] = \int_0^{t_1} \left(\frac{m}{2}\left(\dot{x}_0^2 + \dot{\xi}^2\right) - \frac{k}{2}\left(x_0^2 + \xi^2\right) \right) dt$$

Hint: you will have cross terms involving x_0, ξ, and their first derivatives. Use integration by parts and the fact that x_0 satisfies the equation of motion to eliminate these terms.

b) Let's assume that the true path $x_0(t)$ represents a stationary point in the action. (In fact it is a stationary point, as required by Hamilton's Principle.) What we would like to understand is whether $x_0(t)$ is a minimum in the action or a saddle point in the action. To address this question, we will consider whether the variation $\xi(t)$ increases or decreases the action in the neighborhood of $x_0(t)$. As always, we will only consider fixed time intervals, in this case the time interval from $t = 0$ to $t = t_1$. Let $S_0 =$

$S[\xi = 0]$, the action for the true path, and let $\Delta S = S[\xi] - S_0$, so that ΔS is the change in the action due to variation $\xi(t)$. Then we have

$$\Delta S = S[\xi] - S_0 = \frac{1}{2}\int_0^{t_1} \left(m\dot{\xi}^2 - k\xi^2\right) dt$$

Let's choose a simple triangle function for the variation:

$$\xi(t) = \begin{cases} \dfrac{\varepsilon t}{t_1}, 0 \le t \le \dfrac{t_1}{2} \\ \varepsilon\left(1 - \dfrac{t}{t_1}\right), \dfrac{t_1}{2} \le t \le t_1 \end{cases}$$

Find the condition for t_1 under which ΔS is negative (where the variation has decreased the action), and compare this value of t_1 to the full period of the oscillator. *Remark*: We know that we always *increase* the action around the true path by increasing the kinetic energy term with a high-frequency, small amplitude wiggle. Since the example of the triangle function shows that it is also possible to find variations that *decrease* the action (at least in some situations), this shows that the true path $x_0(t)$ represents a *saddle point* in the action for those cases, not a minimum. In other words, the action may increase or decrease around the true path, depending on the exact nature of the variation that is considered, and depending on the choice of t_1. This is why we should refer to Hamilton's Principle as 'the principle of stationary action' and not 'the principle of least action.'

c) Repeat part b) for a variation of the type $\xi(t) = \varepsilon \sin(\pi t/t_1)$.

Extra Credit

1) Problem 8.14 Orbits with different central force laws